Series Review - Part I

Convergence tests for series of constants

① Sequences converge if $\lim_{n\to\infty} a_n = L$. There are six sequences that occur often & bear memorizing.

a) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ b) $\lim_{n\to\infty} \sqrt[n]{n} = 1$ c) $\lim_{n\to\infty} x^{1/n} = 1 \quad x>0$

d) $\lim_{n\to\infty} x^n = 0 \quad |x|<1$ e) $\lim_{n\to\infty} (1+\frac{x}{n})^n = e^x$ f) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

② Series convergence

Definition: For the infinite series $\sum a_n$, the n^{th} partial sum

$S_n = a_1 + a_2 + a_3 + \dots + a_n$.

If the sequence of partial sums $\{S_n\} = \{S_1, S_2, S_3, \dots, S_n\}$ converges, the series is said to converge to the same number.

The definition is most useful with telescoping series. It can also be demonstrated with the geometric series.

Geometric Series $\sum_{n=0}^{\infty} ar^n$ Example $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$ the series converges to $\frac{a}{1-r}$

If $|r|<1$ $(-1<r<1)$,

N^{th} term test for divergence Given the series $\sum a_n$, if $\lim_{n\to\infty} a_n \neq 0$, the series diverges. (Note: If $\lim_{n\to\infty} a_n = 0$, you have no information.)

Integral Test - If f is positive, continuous, and decreasing for $x \geq 1$ and if $a_n = f(n)$, then

$\sum a_n$ and $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

P-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ converges when $p>1$

It diverges when $0<p\leq 1$

Harmonic series - is a p-series where $p=1$

$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ It is divergent.

Direct Comparison Test

Let $0 \le a_n \le b_n$ for all n

i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

ii) If $\sum_{n=1}^{\infty} a_n$ diverges, the $\sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ and

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$ where L is finite and positive.

Then the series $\sum a_n$ and $\sum b_n$ converge or diverge together.

Alternating series test. Let $a_n > 0$. Then the alternating series

$\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge if

i) $\lim_{n\to\infty} a_n = 0$ and ii) $a_{n+1} \le a_n$

This is also called the Leibniz Theorem.

The alternating series Remainder. – The sum S of a convergent alternating series can be estimated by S_n (the n^{th} partial sum). The maximum error in using this estimation (or the remainder) is less than the first unused term.

$$|S - S_n| = |R_n| \le a_{n+1}$$

Absolute & Conditional convergence

If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

$\sum a_n$ is said to be absolutely convergent.

A series $\sum a_n$ is said to be conditionally convergent if $\sum |a_n|$ diverges and $\sum a_n$ converges.

The terms of an absolutely convergent series can be re-arranged. The terms of a conditionally convergent series cannot.

Series Review - Part II

Power series

Taylor Series If f is a function with derivatives of all orders at $x=c$ then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c)+f'(c)(x-c)+\dots f''\frac{(c)(x-c)^2}{2!}+\dots$$

A MacLaurin Series is a Taylor series centered at zero.

The Remainder (error) when estimating the sum of a Taylor Series by a Taylor Polynomial is given below.

If $f(x)$ is estimated by the n^{th} order Taylor Polynomial

$$f(c)+f'(c)(x-c)+f''\frac{(c)(x-c)^2}{2!}+\dots+f^{(n)}\frac{(c)(x-c)^n}{n!}$$

then the remainder (error) is given by

$$R_n(x) = \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!} \text{ where } c<z<x \text{ (or } x<z<c\text{)}$$

Geometric Power Series If a function f can be manipulated to fit the form $\frac{a}{1-r}$, then it can be expressed as the geometric series $\sum_{n=0}^{\infty} a(r)^n$.

Ex: $\frac{1}{1+x} = \frac{1}{1-(-x)}$ $\frac{1}{1+x} = \sum_{n=0}^{\infty} 1(-x)^n$

Manipulation of Series

From a given series, new series can be created by substituting something for x, by multiplying or dividing by a constant or a variable, by differentiating or integrating term by term.

Ex. $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$

$$e^{x^2} = 1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\dots$$

$$\frac{1}{2}e^{x^2} = \frac{1}{2}+\frac{x^2}{2}+\frac{x^4}{2\cdot 2!}+\frac{x^6}{2\cdot 3!}+\dots$$

The Ratio Test - Given $\sum a_n$ if:

① $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, $\sum a_n$ converges

② $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $= \infty$, $\sum a_n$ diverges

③ $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the test fails

The Root Test - Given $\sum a_n$ with non-zero terms if:

① $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, $\sum a_n$ converges

② $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $= \infty$, $\sum a_n$ diverges

③ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the test fails.

Radius of Convergence

The radius of convergence is the distance from the center of the series to either endpoint of the interval of convergence. The endpoints are not considered.

Ex $\sum_{n=0}^{\infty} 3(x-2)^n \qquad \lim_{n\to\infty} \left| \frac{3(x-2)^{n+1}}{3(x-2)^n} \right| = \lim_{n\to\infty} |x-2|$

$$-1 < x-2 < 1 \qquad R = 1$$

When the interval of convergence is $\mathbb{R}$, all real numbers, the radius of convergence is $R = \infty$.
When the interval of convergence is only the center of the series $R=0$

Interval of convergence after integrating or differentiating a series.

If a new series is created by differentiating or integrating a series, the interval of convergence remains the same except, perhaps, at the endpoints. Hence, only the endpoints need to be checked after differentiation or integration.

Intervals of Convergence

To find the interval of convergence of a series, first apply the ratio test. If the limit is <1, the interval of convergence is $\mathbb{R}$. If the limit is >1, the interval of convergence is c, the center of the series. If the limit contains x, solve $-1 < \text{limit} < 1$ for the interval of convergence.

Since the ratio test fails at limit $=1$, the endpoints of the interval of convergence must be checked separately.

Since a geometric series diverges when $r=1$ and $r=-1$, the endpoints do not have to be checked for a geometric power series.

Ex. Find the interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}}}{\frac{(-1)^{n+1}(x-5)^n}{n5^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}\cdot n\cdot 5^n}{(x-5)^n(n+1)5^{n+1}}\right| =$$

$$\lim_{n\to\infty}\left|\frac{(x-5)\,n}{(n+1)\cdot 5}\right| = \left|\frac{x-5}{5}\right|$$

$$-1<\frac{x-5}{5}<1$$

$$-5<x-5<5$$

$$0<x<10$$

check endpoints.

$x=10$ $\quad \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(10-5)^n}{n\cdot 5^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}5^n}{n\cdot 5^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

Alternating harmonic series, convergent

$x=0$ $\quad \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(0-5)^n}{n\cdot 5^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n(5)^n}{n\cdot 5^n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}$

(-1) times the harmonic series, divergent.

Interval of convergence is $(0, 10]$

Series you should know.

$$\frac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots+(-1)^n(x-1)^n+\cdots \qquad 0<x<2$$

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots+x^n+\cdots \qquad -1<x<1$$

$$\frac{1}{1+x} = 1-x+x^2-x^3+x^4-x^5+\cdots+(-1)^n x^n+\cdots \qquad -1<x<1$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^n x^{n+1}}{n+1}+\cdots \qquad -1<x<1$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\cdots \qquad -\infty<x<\infty$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots \qquad -\infty<x<\infty$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots \qquad -\infty<x<\infty$$